Rolling horizon production planning for probabilistic time-varying demands

JAMES H. BOOKBINDER† and BAK-TEE H'NG†

This paper introduces and tests a production planning procedure to be applied in a rolling horizon with probabilistic demands, based on the work of Bookbinder and Tan (1983). First, the case of no lead time for replenishment is considered and trend-seasonal demands are studied. We vary the set-up cost, order cycle and number of periods in the future for which demand forecasts are available. The procedure is compared to that of Silver (1978) in terms of cost performance, percentage of demand short/period and percentage of periods with stock-outs. Each approach appears to have merit. The proposed procedure generally yields a lower solution cost, while Silver's procedure usually has a lower percentage of demand short/period and a smaller percentage of periods with stockouts. Differences between the two procedures were relatively insensitive to which of three lot-sizing rules was employed within them. Finally, we further extended our procedure to consider non-zero lead times for replenishment and different demand patterns, particularly the normal distribution.

Introduction

A number of recent papers have studied lot-sizing in a rolling horizon environment when demands are deterministic (Baker 1977, Blackburn and Millen 1980, Chand 1982, Bookbinder and H'ng 1986). In practice, however, future demands are often uncertain. Several approaches to solve the probabilistic problem have thus appeared in the literature.

Silver (1978) presented a procedure to determine the timing and sizes of replenishments for probabilistic, time-varying demand when forecast errors are normally distributed. Production occurs when the current inventory position cannot provide the desired safety factor based on demand variability during the lead time. This procedure attempts to satisfy an integer number of periods' requirements, using e.g. the Silver–Meal heuristic (1973) (see also Silver and Peterson 1985). The production quantity must include a buffer stock because of the stochastic nature of demand.

An alternative approach, that of Askin (1981), determines lot-sizes and replenishment epochs for dynamic probabilistic demand based on least cost per unit time. (The mean and variance of demand, and any demand correlations, are assumed known over the horizon.) The optimal order-up-to level is found for the desired cycle

---

Revision received November 1985.
† Department of Management Sciences, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.
length $T_0$, and minimal total cost is achieved when the order in fact lasts for $T_0$
periods.

Similar to most heuristics methods, neither of the above procedures can
guarantee optimality. The extensive computational requirements to obtain an
optimal solution to the probabilistic dynamic lot-sizing problem rule out the
possibility of implementing it (Silver 1978).

Another method for this problem (Bookbinder and Tan 1983) minimizes
expected cost subject to an upper bound constraint on the probability of stockout
during any period. The procedure proposed in the present paper will be based upon
that research and applied in the context of a rolling schedule.

In an approach distinct from those above, Wemmerlov and Whybark (1984)
studied 14 different single-stage lot-sizing methods in a rolling schedule environment
when demands are probabilistic. Cost performances were compared under 100% service levels. Factors such as the order cycle, coefficient of variation of demand, lead
time and demand uncertainty were varied. Actual demand for any period was the
simulated forecast demand minus a simulated forecast error generated from that period's error distribution. Their findings rank the lot-sizing methods differently in
the cases of uncertainty and no uncertainty in demands. When demand is uncertain,
no significant differences were found between the six lot-sizing methods which were
best for deterministic demands in a rolling horizon. (See also DeBoodt and Van
Wassenhove 1983 a, b).

The work of Wemmerlov and Whybark is important in that, for the first time, a
rolling horizon study was conducted for the case of probabilistic demands. The
emphasis of the present paper differs in the following ways. First, our work
concentrates on the procedure for probabilistic production planning, rather than the
lot-sizing rule employed within that procedure. Demand forecasting is thus a part of
our methodology, and this has been applied to various demand patterns. Secondly,
we have studied the effectiveness of the planning procedures as a function of the
forecast window $M$, the number of periods in the future for which demand estimates
are available. Finally, several measures of customer service are important output
variables in the present paper.

In the coming sections, statements of the probabilistic lot-sizing problem, the
assumptions made and the objective function will be presented. The proposed
procedure to solve this stochastic problem in a rolling horizon environment will be
discussed.

Throughout this paper, the term 'procedure' will refer to a general framework for
production planning under stochastic demand. 'Silver's procedure', for example,
should thus not be confused with the Silver–Meal (1973) lot-sizing heuristic. In fact,
during the course of this research, several different lot-sizing heuristics were
embedded in the probabilistic procedures. These include the Wagner–Whitin
algorithm (1958), Heuristic 2 of Bookbinder and Tan (1985) and the Silver–Meal
heuristic.

Research to date on the probabilistic problem (Silver 1978, Akin 1981,
Bookbinder and Tan 1983, Wemmerlov and Whybark 1984) has not included an
in-depth comparison of production planning procedures. Our stochastic procedure will
be compared to that of Silver (1978). Details of parameters to be varied for the three
demand patterns and two probabilistic procedures are later presented, followed by
numerical results and conclusions. (More extensive calculations are contained in
H'ng 1984.)
Assumptions

To develop an uncapacitated single-item production schedule in a rolling horizon environment, when there is stochastic demand and a constraint on the probability of stockout in any period, we assume the following:

1. Demand for the product is probabilistic and seasonal in nature. Some historical information is available at the start of the planning horizon.
2. A fixed cost is incurred for each set-up initiated.
3. The inventory carrying cost is proportional to the inventory on hand at the end of each period.
4. The lead time for replenishments is zero (later to be generalized).
5. The variable cost per unit of production is constant.
6. Quantity produced in any period is available at the start of that period.
7. The replenishment quantity can be used to satisfy demand in that period or in later periods.
8. When a stockout occurs, any sales not met will be lost. However, there is a probability \( \alpha \) (set by management) that each period’s inventory be non-negative (\( \alpha \) is the desired service level, say 95%).

The equivalent deterministic problem

To minimize expected total cost incurred in carrying inventory and in set-ups, we must resolve decisions of when to produce, how many periods’ demands this production should cover and, hence, how large the lot-size should be. Bookbinder and Tan (1983) refer to one version of this probabilistic problem as the ‘static uncertainty’ model, for which they were able to obtain an equivalent deterministic problem. Let us briefly review their work, since it motivates the method of the present paper. Recall that the model for the deterministic problem is:

\[
\text{objective function} - \text{minimize total relevant cost (TR')} \]

\[
= \sum_{t=1}^{M} \{ A \cdot \delta(X_t) + h \cdot L_t \} \tag{1}
\]

subject to, for \( t = 1, 2, 3, \ldots, M \)

\[
I_t = I_0 + \sum_{i=1}^{t} (X_i - d_i) \tag{2}
\]

\[
X_t, L_t \geq 0 \tag{3}
\]

\[
\delta(X_t) = \begin{cases} 1 & \text{if } X_t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}
\]

where \( X_t \) is the quantity produced in period \( t \); \( I_t \), inventory carried from period \( t \) to \( t+1; M \), forecast window; \( A \), set-up cost per order; \( h \), cost to carry a unit of inventory from period \( t \) to \( t+1; d_t \), demand in period \( t \); and \( I_0 \), initial inventory.

To obtain a probabilistic model with the above assumptions, Bookbinder and Tan (1983) incorporate a service-level constraint

\[
\Pr\{I_t \geq 0\} \geq \alpha \tag{5}
\]

They introduce the cumulative distribution function \( G_D(d_t) \), where

\[
D(t) = d_1 + d_2 + \ldots + d_t
\]
and its inverse function $G_{\pi_0}^{-1}(\cdot)$ means
\[ \alpha = G_{\pi_0}(u) = \Pr[D(t) \leq u]. \]
Combining eqns. (2) and (5) one has
\[ \sum_{i=1}^{I} X_i \geq G_{\pi_0}^{-1}(\alpha) - I_0. \]

The probabilistic model can thus be mathematically formulated as:

**Objective function** — Min $E[T(R(t))$

\[ \sum_{i=1}^{M} \{ A \cdot \delta(X_i) + h \cdot E[\max(I_i, 0)] \} \]

**Subject to**, for $t = 1, 2, \ldots, M$

\[ E[I_i] = I_0 + \sum_{i=1}^{I} X_i - \sum_{i=1}^{I} E[d_i] \]

\[ \sum_{i=1}^{I} X_i \geq G_{\pi_0}^{-1}(\alpha) - I_0 \]

\[ \delta(X_i) = \begin{cases} 1 & \text{if } X_i > 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ X_i \geq 0 \]

Let us define
\[ \Delta D(t) = G_{\pi_0}^{-1}(\alpha) - \sum_{i=1}^{I} E[d_i] \]

Constraint (7) can be rewritten as
\[ E[I_i] - \Delta D(t) = I_0 + \sum_{i=1}^{I} X_i - G_{\pi_0}^{-1}(\alpha) \]

The service level $\alpha$ is set by management, generally at a fairly high value (we have taken $\alpha = 95\%$ throughout the present paper). The term $E[\max(I_i, 0)]$ may then be approximated by $E[I_i]$. One then observes the following correspondence between the static uncertainty model and the deterministic model:

**Deterministic model**
\[ d_i \]
\[ \sum_{i=1}^{I} d_i \]
\[ I_t \]

**Static uncertainty model**
\[ G_{\pi_0}^{-1}(\alpha) - G_{\pi_0}^{-1}(\alpha) \]
\[ G_{\pi_0}^{-1}(\alpha) \]
\[ E[I_i] - \Delta D(t) \]

Although the deterministic problem does not have a constraint like eqn. (8), this is implicitly included in constraints (2) and (3).

Bookbinder and Tan transformed the static uncertainty model to an ‘equivalent deterministic problem’. This permits the probabilistic problem to be solved by any exact or approximate method for the deterministic lot-sizing problem, applied to the
'equivalent deterministic demands', \( d'_t \). The transformation to the equivalent deterministic problem is as follows:

**Step 1** set \( d'_t = G_{D_0}^{-1}(\alpha) \)

**Step 2** for \( t = 2, 3, \ldots, M \), \( d'_t = G_{D_0}^{-1}(\alpha) - G_{D_0}^{-1}(\alpha) \) is the demand for period \( t \) in the deterministic problem.

### Determination of \( G_{D_0}^{-1}(\alpha) \)

Bookbinder and Tan (1983) (see also Tan (1983)) expressed

\[
G_{D_0}^{-1}(\alpha) = E[D(t)] + z \cdot S[D(t)]
\]

where \( E[D(t)] \) is the mean and \( S[D(t)] \) the standard deviation, assuming a normal distribution of forecast errors. For a desired service level \( \alpha \), the 'safety factor' \( z \) can thus be read from a standardized normal table. To generalize the mathematical structure of the inverse cumulative distribution function, we will consider in the present paper

\[
G_{D_0, y}^{-1}(\alpha) = E[D(t, y)] + z \cdot S[D(t, y)]
\]

The periods 1, 2, \ldots, \( t \) represent months and the 'cycle' \( y \) represents a particular year. \( D(t, y) \) is thus the cumulative demand of months 1 through \( t \) in year \( y \). We assume the monthly demands \( d_{y,t} \) to be independent and identically distributed for each fixed \( t \).

We will use Winters' (1960) exponential smoothing model to forecast the demands \( d_{y,t} \) of months \( j = 1, \ldots, M \) in a given year \( y \). To transform to equivalent deterministic demands, we then employ a 5-period moving average of 4 historical months and 1 future month within that year. These moving averages furnish the estimates \( E[d_{y,t}] \) and \( S[d_{y,t}] \) needed to calculate \( G_{D_0, y}^{-1}(\alpha) \).

The moving average expression for \( E[d_{y,t}] \), the expected demand of month \( j \) in year \( y \), is for \( y \leq 5 \)

\[
E[d_{y,t}] = \left( \sum_{i=1}^{y-1} wt(j, i)d_{y,t} + wt(j, y)d_{y,t} \right) / \sum_{i=1}^{y} wt(j, i)
\]

and for \( y > 5 \)

\[
E[d_{y,t}] = \left( \sum_{i=y-4}^{y-1} wt(j, i)d_{y,t} + wt(j, y)d_{y,t} \right) / \sum_{i=y-4}^{y} wt(j, i)
\]

where \( wt(j, i) \) is the percentage weight given by management to the datum \( d_{y,t} \). The term in \( d_{y,t} \) is included to help estimate a trend in demand. The number of future periods included in the moving average can be increased if the demand forecast is sufficiently accurate. This would require only a minor modification of the procedure to calculate \( G^{-1} \).

The expression for \( y \leq 5 \) is given only for completeness; the start-up phase was eliminated by warming up the model for 60 periods. Reported experimental results do not include calculations from this transient phase. We also remark that for simplicity our formulae assume a 'year' is really the next 12 months, with the time-axis shifted for each new forecast or lot-sizing decision.
Consider now the determination of $S[d_{jy}]$, the standard deviation of demand for month $j$ and year $y$. $S[d_{jy}]$ will be, for $2 \leq y \leq 5$

$$S[d_{jy}] = \left( \left\{ \left[ \sum_{i=1}^{y-1} \frac{(d_{ji} - E[d_{ji}])^2 + (d_{jy} - E[d_{jy}])^2}{(y-1)} \right] \right\}^{1/2} \right)$$

and for $y > 5$

$$S[d_{jy}] = \left( \left\{ \left[ \sum_{i=y-4}^{y-1} \frac{(d_{ji} - E[d_{ji}])^2 + (d_{jy} - E[d_{jy}])^2}{4} \right] \right\}^{1/2} \right)$$

Finally, we have

$$G_{D_{1,y}}(z) = \sum_{j=1}^{t} E[d_{jy}] + z \cdot \left( \sum_{j=1}^{t} S^2[d_{jy}] \right)^{1/2}$$

We remark that all results of the present paper are based upon a service level $\alpha = 95\%$ and moving averages calculated with weights $= 1$.

Design of the probabilistic procedure

Once transformation to equivalent deterministic demand has been carried out (using historical data and forecast demand), any heuristic for the deterministic lot-sizing problem can be used to determine production in period 1. Suppose this first lot intended to satisfy demand for periods 1 through $N$ ($N \leq M$), but in fact covered demand through period $k$. (Note that $k$ need not equal $N$, since the latter is based on equivalent deterministic demand.) Moving the time axis such that period $(k+1)$ becomes period 1, the same lot-sizing problem is again encountered.

Before determining the next production quantity, an update will be made using the demands now known, followed by revised forecasts and changing to equivalent deterministic demands. Because demand is stochastic, there may be a stockout or some inventory remaining at the end of period $k$. The former causes the lot-sizing heuristic no difficulty, while any stock on hand is subtracted from the calculated production quantity. Future lot-sizes will be similarly determined.

As in any stochastic optimization problem, the realized demands may be such that, in hindsight, the lot-sizing decisions were sub-optimal. Nevertheless, the suggested procedure obtains lot-sizes which minimize expected total costs. Moreover, if the desired service level $\alpha$ is increased, the constraint on the probability of no stockout leads to a greater equivalent deterministic demand. The latter is then more likely to exceed actual demand, causing increased total costs. This is the price paid for a higher degree of stockout protection.

Procedure for probabilistic dynamic demand with seasonal nature

Our approach to the probabilistic lot-sizing problem is thus summarized as follows:

1. Use Winters’ model to forecast demands for $M$ periods (or the remaining periods if it reaches the end of the planning horizon) beyond the given historical demands or known demand data.
2. Transform the forecast to equivalent deterministic demands.
Rolling horizon production planning for probabilistic time-varying demands

(3) Use any heuristic for the deterministic problem to obtain the lot-size for the equivalent deterministic demands. If on-hand inventory is positive, the desired production quantity will be the lot-size minus on-hand inventory. Otherwise, it will be the lot-size itself.

(4) As demand unfolds, update the inventory position and the demand forecasting parameters. Forecast demand for the next \( M \) periods (or the remaining periods), changing to equivalent deterministic demands. If on-hand inventory is greater than next period's equivalent deterministic demand, repeat step 4 beginning with updating the inventory position. Otherwise, go to step 3 (stop if the horizon ends).

This procedure will be compared below to that of Silver (1978), a brief description of which is given in the introduction of the present paper.

Description of the experiments

There are three main problem characteristics which may affect a procedure's performance, namely the number of future periods for which demand is known, production set-up and holding costs, and demand patterns. Various combinations will be employed to represent some realistic situations.

We will investigate the effect of the forecast window, cost parameters and the variance of the demand distribution. Forecast windows of 2–12 periods (months) will be used, with demands generated from trend-seasonal cases, i.e. no trend, increasing trend and decreasing trend. The average demand \( D \) is set to 50 units per period. Set-up costs \( A = \$400 \) or \( \$900 \) provide the desired order cycles \( T_0 \) of 4 or 6 months. \( T_0 = \left(\frac{2A}{Bh}\right)^{1/2} \) is the time between orders in an EOQ model with constant demand. \( z = 95\% \) and the holding cost \( h = \$1/\text{unit/month} \) throughout this paper. Eight replications were done on each 300-period demand problem.

Lot-sizing rules

In an earlier paper (Bookbinder and H'ng 1986, see also H'ng 1984) the performances of various lot-sizing techniques, for a rolling schedule where demand is deterministically known, were discussed. Heuristic 2 of Bookbinder and Tan (H2, 1985) and the Silver–Meal heuristic (SM, 1973) were found to be good most of the time for short forecast windows, while the Wagner–Whitin algorithm (WW, 1958) was generally good for long forecast windows.

These three algorithms will therefore be included in the two procedures for probabilistic demand, ours and Silver's (1978). Although there is no guarantee that these heuristics will perform as well here as for deterministic rolling horizon problems, we wished to test mainly the procedure for the probabilistic problem in a rolling horizon, not the lot-sizing heuristic employed. Nevertheless, use of several lot-sizing algorithms allows some comparison with the findings of Wemmerlov and Whybark (1984).

The WW algorithm applies dynamic programming to the deterministic problem and guarantees an optimal static solution. However, it need not be optimal in the rolling horizon. The SM heuristic solves the deterministic lot-sizing problem by minimising the sum of holding and ordering costs per unit time.

Heuristic H2 was shown to perform well in the deterministic static (Bookbinder and Tan 1985) and rolling schedule cases (Bookbinder and H'ng 1986). It attempts to
combine the merits of both SM and least unit cost (DeMatteis and Mendoza 1968) while eliminating their respective drawbacks. H2 is depicted as follows:

Step 1 Let \( N = 1, T_1 = 1, D_1 = d_1, H_1 = 0, F_1 = A \)
Step 2 Let \( N = N + 1 \)
Step 3 3(a) If \( d_N = 0 \), go to Step 3 (b)
   Otherwise, go to Step 3 (c)
   3(b) Let \( T_N = T_{N-1} \) and go to Step 4
   3(c) Let \( T_N = T_{N-1} + 1 \) and go to Step 4
Step 4 4(a) Let \( D_N = D_{N-1} + d_N \)
   4(b) Let \( H_N = H_{N-1} + h[(N - 1)/T_N]d_N \cdot D_N \)
   4(c) Let \( F_N = A/T_N + H_N/D_N \)
Step 5 If \( F_N > F_{N-1} \), go to Step 6
   Otherwise, go to Step 2
Step 6 Let the lot size be \( D_{N-1} \) and Stop.

Experimental results

Comparison using realized demand

No optimal solution method is available for the case of probabilistic demand in a rolling horizon. Once the randomly-generated demands are known, however, the static 300-period realized-demand problem may be solved using the Wagner–Whitin algorithm to obtain a lower bound on the solution. The total rolling schedule cost of our probabilistic procedure and that of Silver (1978) are each expressed as percentage differences from this lower bound, referred to below as ‘cost deviations’.

That may seem a very crude measure, since each procedure is actually solving stochastic demand problems (in a rolling horizon of at most 12 periods in length). Both procedures are at an equal disadvantage, however, and although the percentage cost deviation for each may be large, it is the relative performance of the two procedures that is of interest. (That is, the procedure with the smaller percentage cost deviation has lower total relevant costs.) Additional bases of comparison are furnished by the service measures: percentage of periods with stockouts and the percentage of demand short per period.

Our experiments studied demand patterns of the trend-seasonal type: no trend, increasing trend and decreasing trend. These reflect demand variations commonly encountered in practice. The outputs of eight 300-period relocations were averaged to obtain each of Figs. 1–6. These six cases correspond to the three demand patterns and set-up costs \( A = $400 \) or \( $900 \). For seasonal no-trend demand, we employed SM and WW within each of the two procedures. H2 and WW were embedded in both procedures for the other two demand patterns.

Trend-seasonal demand (increasing trend)

This model is of the form

\[ d_t = (12.4 + 0.25t)[1 + \sin(\pi t/6)] + z_t \]

where \( z_t \) is randomly generated from a uniform distribution with mean 0 and range 20. For each set-up cost, our procedure attained the minimum percentage cost deviation, whether it was used with H2 (OPH2) or with WW (OPWW). (See Figs. 1 and 2.) However, there is a marked indication that along with the lower total relevant cost, OPH2 and OPWW resulted in a higher percent of periods with
Figure 1. Results for trend-seasonal (increasing trend), $A = 400$. 
Figure 2. Results for trend-seasonal (increasing trend), A = 900.
stockouts and in greater fraction of demand short per period compared to Silver's procedure with H2 (SPH2) or with WW (SPWW). The choice of procedure thus involves a trade-off of percentage cost deviation with the percent of periods with stockouts and the fraction of demand short/period.

Seasonal demand (no trend)

This model has the form

\[ d_t = 50(1 + \sin(\pi t/6)) + z_t \]

where \( z_t \) is as above. Figures 3 and 4 depict results for a set-up cost of 400 or 900. When \( A = 400 \), our procedure with SM (OPSM) was best compared to OPWW, SPSM.

Figure 3. Results for trend-seasonal (no trend), \( A = 400 \).
and SPWW in terms of total relevant cost. For percentage demand short/period and percentage of periods with stockouts, OPSM and OPWW tend to show better results than SPSM and SPWW for shorter forecast windows (M ≤ 7). When M ≥ 8, Silver’s procedure was superior for these service measures.

For a set-up cost of 900 (Fig. 4), OPSM still yields the smallest percentage deviation. For the percent of periods with stockouts and the fraction of demand short per period, OPSM and OPWW again show good results for small M. As the forecast window increases, SPSM and SPWW often outperform our procedure.
However, these differences in service levels for larger $M$ are not as pronounced as they are when $A = 400$.

Trend-seasonal demand (decreasing trend)

This demand pattern may be expressed as

$$d_i = [12.4 + 0.25(3.01 - t)](1 + \sin(\pi t/6)) + z_i$$

where $z_i$ is again uniformly distributed on $(-10, 10)$. Clearly for $A = 400$ (Fig. 5), OPH2 is the best procedure. It not only attains the minimum percentage deviation from the lower bound, but also the minimum in percent of demand short/period and percent of periods with stockouts. Indeed, for the latter two measures, OPH2 yields

![Graph showing results for trend-seasonal (decreasing trend) $A = 400$.]
100% service for $M \geq 3$. A more detailed interpretation of this and other results is given in the conclusions.

For $A = 900$ and $M \leq 8$, SPH2 has the minimum cost deviation but higher percentages of demand short and periods with stockouts (Fig. 6). For longer forecast windows ($M \geq 9$), OPH2 is best on all three characteristics, namely percentage cost deviation from the lower bound, percent of demand short/period and percent of periods with stockouts.

Figure 6. Results for trend-seasonal (decreasing trend), $A = 900$. 
Comparison of the probabilistic procedures

Our procedure and that of Silver should be evaluated irrespective of the heuristic employed (H2, SM or WW). Thus, we will compare OPH2 to SPH2, and OPSM to SPSM, but not OPH2 to SPSM, etc. Comparisons of the two procedures will be presented in terms of the following for the cases studied

R1 Number of best results (including equal results) for the given procedure, based on total relevant cost
R2 Maximum difference, compared to the other procedure, in cost deviation from the lower bound
R3 Number of best results (including equal results) based on percentage of demand short per period
R4 Maximum difference in percent demand short/period, compared to the other procedure
R5 Number of best results (including equal results) based on percentage of periods with stockouts
R6 Maximum difference in percentage of periods with stockouts, compared to the other procedure

Tables 1 and 2 summarize measures R1–R6 for the various demand patterns, setup costs and lot-sizing heuristics. When \( M = 4 \), e.g., our procedure gave the better result for total cost in 12 instances; in no instance was its cost more than 5.2% further from the lower bound than the cost with SP; OP gave 10 instances of lower percentage of demand short/period, and was never more than 0.09% of demand further short/period than was SP; OP was also best in 10 instances for the percentage of periods with stockouts, and never had more than 0.79% of periods with stockouts beyond that of SP. From Table 2, however, we see that for \( M = 4 \), Silver’s procedure gave the better result for total relevant cost in two instances but its maximum cost deviation from the lower bound was 23.6% above that of OP, etc.

Several comments are in order. The quantities R2, R4 and R6 are more favourable (less unfavourable) when they are smaller. For R1, R3 or R5, the ‘number of best results’ for a given \( M \) need not be constant when summed over

<table>
<thead>
<tr>
<th>( M )</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
<th>R5</th>
<th>R6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>8.3</td>
<td>8</td>
<td>0.25</td>
<td>10</td>
<td>2.17</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>6.3</td>
<td>10</td>
<td>0.08</td>
<td>9</td>
<td>0.79</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>5.2</td>
<td>10</td>
<td>0.09</td>
<td>10</td>
<td>0.79</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>7.9</td>
<td>10</td>
<td>0.10</td>
<td>10</td>
<td>0.92</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>7.4</td>
<td>11</td>
<td>0.05</td>
<td>11</td>
<td>0.54</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>7.8</td>
<td>8</td>
<td>0.09</td>
<td>8</td>
<td>0.67</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>1.1</td>
<td>8</td>
<td>0.09</td>
<td>8</td>
<td>0.79</td>
</tr>
<tr>
<td>9</td>
<td>13</td>
<td>2.2</td>
<td>7</td>
<td>0.07</td>
<td>7</td>
<td>0.58</td>
</tr>
<tr>
<td>10</td>
<td>13</td>
<td>0.2</td>
<td>6</td>
<td>0.06</td>
<td>8</td>
<td>0.54</td>
</tr>
<tr>
<td>11</td>
<td>14</td>
<td>0.0</td>
<td>6</td>
<td>0.07</td>
<td>7</td>
<td>0.58</td>
</tr>
<tr>
<td>12</td>
<td>14</td>
<td>0.0</td>
<td>7</td>
<td>0.06</td>
<td>8</td>
<td>0.54</td>
</tr>
</tbody>
</table>

Table 1. Comparison of our procedure to Silver’s procedure.
<table>
<thead>
<tr>
<th>$M$</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
<th>R5</th>
<th>R6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>104</td>
<td>9</td>
<td>0.05</td>
<td>8</td>
<td>0.71</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>184</td>
<td>5</td>
<td>0.02</td>
<td>5</td>
<td>0.38</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>23.6</td>
<td>4</td>
<td>0.03</td>
<td>4</td>
<td>0.50</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>30.9</td>
<td>4</td>
<td>0.02</td>
<td>5</td>
<td>0.38</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>18.4</td>
<td>4</td>
<td>0.02</td>
<td>5</td>
<td>0.29</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>22.2</td>
<td>6</td>
<td>0.01</td>
<td>6</td>
<td>0.17</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>27.3</td>
<td>7</td>
<td>0.02</td>
<td>6</td>
<td>0.12</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>27.6</td>
<td>8</td>
<td>0.01</td>
<td>9</td>
<td>0.17</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>25.8</td>
<td>8</td>
<td>0.01</td>
<td>8</td>
<td>0.17</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>26.3</td>
<td>9</td>
<td>0.01</td>
<td>9</td>
<td>0.17</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>25.6</td>
<td>7</td>
<td>0.01</td>
<td>7</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Table 2. Comparison of Silver's procedure to our procedure.

Tables 1 and 2. This is because every instance of equal results between the two procedures is counted in both tables. Also, for each set-up cost $A$, any particular best result is an average of eight 300-period replications of a given demand pattern, to avoid undue influence for any of the realized demands.

Our procedure is generally better for R1 and R2, while Silver's procedure is better for R4 and R6. Both procedures thus appear to have merit. These solution results should serve as a guideline to the practitioner of the trade-offs involved in percentage penalty cost, percentage quantity short and percent of periods with stockouts, in deciding whether to use our procedure or Silver's for this problem.

In the next section, we extend our procedure to consider different demand patterns and non-zero lead times for replenishment.

**Extensions of our procedure**

**Different demand patterns**

Our procedure will allow other distributions beside trend-seasonal cases if we modify the determination of $G^{-1}$. Suppose that demands in different months are independent and identically normally distributed. One obtains for a given year, for $2 \leq j \leq 5$,

$$E[d_j] = \left\{ \frac{\sum_{n=1}^{j-1} [wt(n)d_n] + wt(j)d_j}{\sum_{n=1}^{j} wt(n)} \right\}$$

$$S[d_j] = \left\{ \frac{\sum_{n=1}^{j-1} (d_n - E[d_n])^2 + (d_j - E[d_j])^2}{(j-1)} \right\}^{1/2}$$

and for $j > 5$,

$$E[d_j] = \left\{ \frac{\sum_{n=j-4}^{j-1} [wt(n)d_n] + wt(j)d_j}{\sum_{n=j-4}^{j} wt(n)} \right\}$$

$$S[d_j] = \left\{ \frac{\sum_{n=j-4}^{j-1} (d_n - E[d_n])^2 + (d_j - E[d_j])^2}{4} \right\}^{1/2}$$
One then has
\[ G_{D01}(a) = \sum_{t=1}^{n} E[d_i] + z \left( \sum_{t=1}^{n} S^2[d_i] \right)^{1/2} \]

With this expression for \( G^{-1} \), lot-size calculations follow as before, with production again based on the latest equivalent deterministic demands.

**Lead time for replenishment**

We previously assumed that the lead time for production or replenishment is zero. To extend our model, we now include a fixed, known replenishment lead time \( L \). The entire production quantity ordered at time \((j - L)\) is assumed to be available at the beginning of period \( j \). To consider (possibly) non-integer \( L \), we modify the rule of when and how much to produce.

Let us first study the cases \( 0 \leq L < 1 \) and \( 1 \leq L < 2 \), and then try to formulate the rule in general. For instance, when \( 0 \leq L < 1 \) (Fig. 7(a)), the review at time \( k - L \) must decide whether inventory can last until \( k + 1 - L \) (the next review) and the arrival at \( k + 1 \) of that next lot. To do so, stock on hand must be able to cover equivalent deterministic demand for periods \( k - L \) through \( k + 1 \) with probability \( a \).

Our ordering policy for \( 0 \leq L < 1 \) will thus be as follows: place an order at time \( k - L \) if on-hand inventory \( Y(k, L) = Ld_{k-1} + d_k \). (This assumes that demand occurs at a constant rate during any given period.) Here \( d_{k-1} \) and \( d_k \) are equivalent deterministic demands for periods \( k - 1 \) and \( k \), respectively. If on-hand inventory at a review instant is \( \geq Y(k, L) \), then no order is placed. That inequality indicates

---

**Figure 7.** Time sequence of placement and receipt of orders. (a) \( 0 \leq L < 1 \). (b) \( 1 \leq L < 2 \). (\( e \rightarrow f \) means order placed at period \( e \) arrives at period \( f \).)
sufficient stock to cover demand until the next review and until any lot-size decided at that time arrives. \( Y(k, L) \) thus plays the role of a dynamic reorder point.

Let us next turn attention to the case \( 1 \leq L \leq 2 \) (Fig. 7(b)). Now at a particular review, we face the question of whether present inventory plus the most recently decided lot size can cover demand until the next review and until this lot arrives. Similar to the above argument, we would place an order if the stock on hand
\( < Y(k, L) \), but the function \( Y \) must now be defined slightly differently
\[
Y(k, L) = L_k d_{k-2} + d_{k-1} + d_k
\]
where \( L_k \) refers to the decimal part of \( L \).

The two specific cases for different ranges of \( L \) suggest the following generalization. For \( l \leq L < l + 1 \), the function \( Y \) shall be defined as
\[
Y(k, L) = L_k d_{k-l-1} + d_{k-l} + \ldots + d_k
\]
The production quantity, as before, is based on the lot size for equivalent deterministic demands. Note that when the replenishment lead time \( L = 0 \), we would place an order if on-hand inventory \( < Y(k, 0) = d_k \), in agreement with our earlier results.

Conclusions

This paper has considered an important problem in production planning and inventory control, that of choosing single-item lot-sizes in a rolling schedule with probabilistic demands. The model (eqns. (6)–(10)) minimizes the expected set-up plus holding costs. Two procedures for this problem were studied. Our procedure was found to be better than that of Silver (1978) in terms of percentage cost deviation from the lower bound. However, Silver's procedure is superior based on percentage of demand short/period and percentage of periods with stockouts.

It may be argued that results achieved by the probabilistic procedures can only be as good as the forecasting methodology employed. Naturally, both procedures had to operate with identical forecasts. In all cases, the two procedures satisfied at least 95% of demand from stock and less than 5% of periods had stockouts. The forecasting methodology does not appear to have impeded the procedures’ performance.

A comment is in order concerning the realized percentage of periods with stockouts, which would be expected to be close to the targeted value of \( (1 - \alpha) = 5\% \). If one solved many static, \( M \)-period problems (eqns. (6)–(10)), 5% of periods would have stockouts in the long run. For a rolling horizon environment, however, one imposes the constraints (5) or (8) more often than once every \( M \) periods. Thus, the first lot size very likely needs to cover less than \( M \) periods of demand, and as a result, fewer than 5% of periods will have stockouts if the forecasting procedure is good.

Results for the two probabilistic procedures were relatively insensitive to which of three lot-sizing heuristics was embedded in them. This is consistent with findings of Wemmerlov and Whybark (1984) and of DeBodt and Van Wassenhove (1983a), each of which reported insignificant differences in solution costs when various lot-sizing heuristics were employed in the case of uncertainty in demand.

For smaller forecast windows \( M \), the cost deviation from the lower bound is a decreasing function of \( M \) for each value of \( A \) and the three demand patterns (Figs. 1–6). The production plan is better when there is more information, even if this information includes some uncertainty.
In the deterministic case (Baker 1977, Blackburn and Millen 1980, Bookbinder and H'ng 1986), one generally requires a window $M \geq T_0$ for an acceptable rolling schedule solution. Results for the probabilistic problem indicate that both customer service measures are quite satisfactory even for $M = 2$. Our procedure and Silver's each develop production plans based on forecast demands, while $T_0$ (= 4 or 6) is calculated from realized demands. The natural cycle $T_0$ thus does not seem to have as much significance for probabilistic as it does for deterministic rolling schedules.

The effect of the demand pattern may be seen in the percentage of periods with stockouts. Both procedures generally yield best results for seasonal demand (decreasing trend), followed by seasonal (no trend), while seasonal (increasing trend) is the least favorable demand pattern. This may be understood by noting that the forecasting method must first identify the presence of a trend before the production plan can react. If one 'under-produces' in the case of a decreasing trend, there is little chance of a stockout while, unless one 'over-produces' for an increasing trend, there is a good chance of a stockout.

Extensions of our procedure to non-zero lead time for replenishment and normally distributed demands were presented. These extensions remain to be tested numerically.

Acknowledgments

This research was supported by the Natural Sciences & Engineering Research Council of Canada, Grant A5292. We are grateful to an anonymous referee for an extremely thorough review of an earlier draft of this paper.

Cet article présente et examine une procédure de planification de la production applicable à un horizon en déroulement avec des demandes probabilistes, sur la base des travaux de Bookbinder and Tan (1983). Premièrement, l'absence de décalage pour réapprovisionnement est examinée et les demandes à tendance et saisonsnières étudiées. Nous faisons varier les coûts d'installation, le cycle de commandes et le nombre de périodes futures pour lesquelles il existe des prévisions de la demande. Cette procédure est comparée à celle de Silver (1978) en termes de rapport rendement/coût, de pourcentage de manque de demande/ périodes et de pourcentage de périodes avec insuffisance de stocks.

Ces approches présentent chacune leurs avantages. La procédure suggérée fournit généralement une solution plus économique alors que la procédure de Silver permet habituellement un pourcentage inférieur manque de demande/ période et un pourcentage réduit de périodes avec insuffisance de stocks. Les écarts enregistrés entre les deux procédures étaient relativement peu sensibles au fait que l'une de trois règles de classement des lots fut appliquée plutôt qu'une autre. Et enfin, nous avons étendu notre procédure à l'examen de décalages non équivalents à zéro pour le réapprovisionnement et de tendances différentes de la demande, notamment la distribution normale.

In diesen Beitrag wird ein Fertigungsplanungsverfahren eingeführt und geprüft, das für die Anwendung auf einen Rollhorizont mit Wahrscheinlichkeitsannahme gedacht ist und auf der Arbeit von Bookbinder and Tan (1983) basiert. Zunächst wird für den Fall, daß die Bestandsergänzung keine Vorbereitungszeit erfordert, die trend-/saisonbedingte Nachfrage untersucht. Dabei werden die Einrichtiekosten, der Bestellzyklus und die zukünftige Periodenzahl, für die Nachfrages-undervorhersagen verfügbar sind, variiert. Das Verfahren wird in bezug auf das Kosten-Leistungs-Verhältnis, den Prozentsatz unzufriedener Nachfrage pro Periode und den Prozentsatz der Perioden mit
Rolling horizon production planning for probabilistic time-varying demands


References


DeBodt, M. A., and Van Wassenhove, L. N., 1983 b, Lot sizes and safety stocks in MRP: A case study, Production and Inventory Management, 24, 1.


H’ng, B. T., 1984, Rolling horizon procedures for deterministic and probabilistic production planning, MSc. Thesis. Department of Management Science, University of Waterloo, Ontario, Canada.

Silver, E., 1978, Inventory control under a probabilistic, time-varying demand pattern, AIE Transactions, 10, 371.

Silver, E., and Meal, H. A., 1973, A heuristic for selecting lot size requirements for the case of a deterministic time-varying demand rate and discrete opportunities for replenishment, Production and Inventory Management, 14, 64.


Winters, P. E., 1960, Forecasting sales by exponentially weighted moving averages, Management Science, 6, 324.