Vehicle routing considerations in distribution system design

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Theory and Methodology

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Abstract: In Distribution System Design, one minimizes total costs related to the number, locations and sizes of warehouses, and the assignment of warehouses to customers. The resulting system, while optimal in a strategic sense, may not be the best choice if operational aspects such as vehicle routing are also considered.

We formulate a multi-commodity, capacitated distribution planning model as a non-linear, mixed integer program. Distribution from factories to customers is two-staged via depots (warehouses) whose number and location must be chosen. Vehicle routes from depots to customers are established by considering the “fleet size and mix” problem, which also incorporates strategic decisions on fleet makeup and vehicle numbers of each type. This problem is solved as a generalized assignment problem, within an algorithm for the overall distribution/routing problem that is based on Benders decomposition. We furnish two versions of our algorithm denoted Technique I and II. The latter is an enhancement of the former and is employed at the user’s discretion. Computer solution of test problems is discussed.

Keywords: Distribution, location, vehicle routing

1. Introduction

Physical distribution presents challenges, both for management and for operational research, because it consists of a number of functional departments (transportation, inventory, warehousing, etc.), each managed somewhat autonomously. Cost/Benefit tradeoffs across organizational boundaries are not straightforward. A familiar example is the reluctance of a traffic manager to incur increased transportation expenses, even if that makes possible still greater savings in inventory costs. Nevertheless, attainment of corporate goals requires coordination or integration of distribution decisions.

This paper involves coordination of decisions which concern the traffic manager (vehicle routing; the number and sizes of vehicles in the fleet), the manager of warehousing (number, location and sizes of warehouses), and the inventory manager (product mix at each warehouse). Interrelated issues such as these should be coordinated by the Vice President of Physical Distribution. We believe the model formulated and solved in this paper could be an important aid in coordinating these logistics functions. Section 2 reviews previous research on physical distribution and vehicle

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routing problems, while Section 3 describes our combined distribution/routing model. Succeeding sections discuss the solution techniques developed; results of test problems; and conclusions and suggestions for further research.

2. Physical distribution and vehicle routing models: A review

2.1. Physical distribution models

The simple plant location problem (SPLP) concerns the location of uncapacitated plants or facilities so as to minimize the total cost of serving customers, i.e., the fixed and variable costs of each facility and the transportation cost. Given a set of customers with known demands for a single commodity, and direct transportation routes from facilities to customers, one finds a minimum cost plan indicating the number of open facilities, their location and the amount shipped from the facility to each customer. An extensive review of the SPLP can be found in Krarup and Pruzan (1983).

Several extensions to the SPLP over the years have included multi-staged, capacity, multicommodity and dynamic features, and piece-wise linear costs. (See Aikens (1985) for a review). Solution strategies for distribution models can be classified as one of the following approaches: mathematical programming (Jasinska and Wojtynych, 1984); exact procedures (Akin and Khumawala, 1977); simulation (Bookbinder, 1984); heuristics (Eldredge, 1982); and decomposition (Geoffrion and Graves, 1974).

In the latter, the most comprehensive work on distribution design, Geoffrion and Graves applied Benders partitioning and decomposed a multi-product problem into a series of single-product problems. The objective function consists of a linear supply transportation cost, the variable throughput cost and fixed cost of each distribution centre (warehouse or depot). (See also Geoffrion, Graves and Lee, 1982).

2.2. Vehicle routing models

In the basic vehicle routing problem, a set of routes begin and end at a central depot and service all nodes so that total distance travelled is minimized. Each route is thus a sequence of nodes that a vehicle visits. We assume there are no temporal restrictions, demand at each node is deterministic and each vehicle has a known capacity.

In the multi-depot, multi-vehicle routing problem, the fleet of vehicles must now serve more than one depot. A constraint that each vehicle must leave from and return to the same depot is added. Extensive references are given in Bodin et al. (1983) and Golden and Assad (1986).

2.3. Routing–allocation models

Incorporating vehicle routing into strategic distribution plans can achieve considerable savings in distribution costs and improve customer service. The literature contains only a few papers on the combined routing–allocation problem.

In a two-staged distribution system, products are sent from factories via depots to customers. Most of the applications have considered the tour pattern of Figure 1 in which several customers are served on one route outgoing from a depot. The decisions made in solving this combined problem are the number and location of depots and the design of tours originating at the depots to serve customers. The solution methods for the routing-allocation problem fall into four broad categories: allocate first-route second (Or and Pierskalla, 1979); route first-allocate second (Perl and Daskin, 1985); tour construction heuristics (Jacobsen and Madsen, 1980) and exact relaxation algorithms (Laporte et al. 1981, 1983). The first two methods typically use good heuristic solutions to the location and routing problems. (See also Nambiar et al. (1981) and the survey by Madsen (1981) and Laporte (1988)).

These approaches were generally quite successful for the particular problem application. The
focus of the present paper, however, is different
directed to a case not covered in the cited publica-
tions. For example, only Laporte et al. (1981,
1983) furnish an 'exact' algorithm rather than a
heuristic. More than one stage of distribution is
considered only by Jacobsen and Madsen, by
Nambari et al and by Or and Pierskalla. All six
papers treated the case of a single product or
'standard product mix' only.

3. Distribution system design / vehicle routing
model

We consider below a multi-product problem,
for two stages of distribution (factories—
warehouses—customers). An 'exact' algorithm is
furnished, combining the Geoffrion and Graves
(1974) approach to distribution system design with
the Fisher and Jaikumar (1981) setting of the
vehicle routing problem, but in fact treated here as
a fleet size and mix problem (Golden et al., 1984).
The Geoffrion—Graves and the Fisher—Jaikumar
approaches are each highly successful in their own
domain. It thus appeared that a model for distri-
bution system design which incorporates vehicle
routing could build on these two techniques.

Our combined distribution/routing model dif-
fers in two respects from the mixed-integer pro-
Introduction of vehicle routing causes the dis-
bution problem to become non-linear in both the
objective function and in some of the constraints,
and also changes the way in which outbound
transportation costs are derived.

Our solution technique will still resemble that
of Geoffrion and Graves, based on Benders de-
composition but modified as in Figure 2. After
solving the master problem (henceforth called the
'warehouse problem'), the vehicle routing problem
for each open distribution centre is solved. Since
the warehouse problem indicates which distribu-
tion centres are open and the assignment of
customers to these warehouses, all necessary in-
formation is provided to solve the routing problem.

That problem is formulated as a fleet size and
mix problem (Golden et al., 1984). Its solution
yields not only the route for each vehicle, but also
the vehicle type (capacity) on which each customer
should be served and the number of each vehicle
type required at the distribution centres. The gen-
eralized assignment approach of Fisher and
Jaikumar (1981) for the solution of the vehicle
routing problem is adapted to include the fleet
size and mix problem. Solution of this routing
problem provides the outbound costs. The trans-
portation sub-problems are then solved as per the
Geoffrion and Graves technique.

Combining the distribution model with vehicle
routing couples the two problems indirectly (but
exactly) via the algorithm of Figure 2. (The distri-
bution problem could have been expanded to di-
rectly incorporate the vehicle routing variables,
but with a greater increase in both the number of
binary variables and constraints.)

To discuss our model in detail, recall the Geo-
frion—Graves distribution formulation. The indices
(i, j, k, l) denote (product, plant, possible distri-
bution centre, customer).

Coefficients

\[
\begin{align*}
S_{ij} & = \text{supply/production capacity of plant } j \text{ for product } i, \\
V_{min_k}, V_{max_k} & = \text{minimum (maximum) throughput for distribution centre } k, \\
D_{il} & = \text{demand for product } i \text{ by customer } l, \\
F_k & = \text{fixed cost of distribution centre } k, \\
V_{ik} & = \text{variable unit cost of throughput of product } i \text{ through distribution centre } k, \\
B_i & = \text{volume per unit that product } i \text{ takes up,} \\
c_{ijkl} & = \text{average unit cost of producing and shipping product } i \text{ from plant } j \text{ through distribution centre } k \text{ to customer } l.
\end{align*}
\]

Variables

\[
x_{ijkl} = \text{flow of product } i \text{ from plant } j \text{ through distribution centre } k \text{ to customer } l, \\
y_{kl} = (0, 1) \text{ variable that will be 1 if distribution centre } k \text{ serves customer } l, 0 \text{ otherwise,} \\
z_k = (0, 1) \text{ variable that will be 1 if distribution centre is open at site } k, 0 \text{ otherwise.}
\]

Min
\[
\begin{align*}
\sum_{i} \sum_{j} \sum_{k} \sum_{l} c_{ijkl} x_{ijkl} + \sum_{k} F_k^2 + \sum_{i} \sum_{k} \sum_{l} V_{ik} D_{il} y_{kl}, \\
\end{align*}
\]

(1)
J.H. Bookbinder, K.E. Reece / Distribution system design

s.t.  \[ \sum_{k} x_{ijkl} \leq S_{ij}, \quad \forall i, j, \]  \[ \sum_{j} x_{ijkl} = D_{il} y_{kl}, \quad \forall i, k, l, \]  \[ \sum_{k} y_{kl} = 1, \quad \forall i, \]  \[ y_{kl} \leq z_{k}, \quad \forall k, l, \]  \[ \forall i, k, l, \] \[ V_{\min} \leq \sum_{k} \sum_{l} B_{il} D_{il} y_{kl} \leq \forall \forall k, \]  \[ (2) \]  \[ (3) \]  \[ (4) \]  \[ (5) \]  \[ (6) \]

Linear configuration constraints
on y and/or z,
\[ y_{kl} = 0 \text{ or } 1, \quad z_{k} = 0 \text{ or } 1, \]  \[ x_{ijkl} \geq 0. \] \[ (7) \]  \[ (8) \]  \[ (9) \]

4. Solution technique

This formulation forces a customer to be serviced by a single warehouse but allows solution by Benders decomposition. That technique applied to (1) through (9) is the basis of our algorithm (Figure 2). The routing problem and the determination of outbound transportation costs have been inserted before solving the transportation subproblems.

**Step 0.** Set \( H = 0 \) (\( H \) denotes the iteration number). If the binary array \((y^1, z^1)\) satisfies (4) to (7), then go to Step 2, otherwise go to Step 1. Set UB = \(-\infty\) and LB = \(-\infty\).

![Figure 2. Solution technique I](image)

**Step 1.** Solve the warehouse problem

Min  \[ \sum_{k} F_{k} z_{k} + \sum_{i} \sum_{l} \sum_{k} V_{ik} D_{il} y_{kl} + Y_{0}, \]  \[ \text{s.t.} \]  \[ (4) \]  \[ (7), (9), \]

\[- Y_{0} + \sum_{i} \sum_{k} P_{il} D_{il} y_{kl} \]  \[ + \sum_{i} \sum_{j} u_{ij} S_{ij} \leq 0 \]  \[ \text{for } h = 1, \ldots, H. \]

\( Y_{0} \) is the estimate of the transportation cost in the distribution system. For the first iteration, these costs are not known and therefore \( Y_{0} \) is set to zero. Let \((y^{H+1}, z^{H+1}, Y^{H+1})\) be any optimal solution to the above problem. Let \( LB^{H+1} \) be the optimal value of (10).

**Step 2.** Solve the routing problem for each warehouse that is open. Therefore, for each \( k \) for which \( z_{k} = 1 \), the nodes in the routing problem are the warehouse location, denoted \( 0_{k} \), and the customers that were temporarily assigned to that warehouse, i.e., customers \( l \) for which \( y_{kl} = 1 \).

Our vehicle routing formulation is similar to the Fisher and Jaikumar (1981) generalized assignment model, but with additional constraints so that strategic decisions could be made concerning the number and types of vehicles in the fleet. Indices \( i, m \) denote customers, \( 0_{k} \) the depot and \( t \) the vehicle.

**Coefficients**

\( T \) = number of vehicles,

\( n \) = number of customers assigned to warehouse \( k \); for each \( k, n = \sum_{l} y_{kl} \),

\( b_{t} \) = capacity of vehicle \( t \),

\( f_{t} \) = fixed cost of vehicle \( t \),

\( d_{i} \) = demand by customer \( i = \sum_{l} b_{t_{l}} D_{il} \),

\( c_{ilm} \) = cost of direct travel from customer \( l \) to customer \( m \) by vehicle \( t \).

**Variables**

\( w_{it} = 1 \) if vehicle \( t \) serves customer \( i \); 0 otherwise,

\( v_{ilm} = 1 \) if vehicle \( t \) travels from customer \( i \) directly to customer \( m \); 0 otherwise.

Define the set \( L_{k} = \{ l: y_{kl} = 1 \} \).
When \( l = 0_k \), this refers to the depot.

\[
\min \sum_{i=0}^{n} \sum_{m=0}^{n} \sum_{t=1}^{T} c_{imt} v_{imt} + \sum_{t=1}^{T} \sum_{i=0}^{n} f_i w_{0it} \\
\text{s.t. } \sum_{i=1}^{n} d_i w_{it} \leq b_i, \forall t, \\
\sum_{t=1}^{T} w_{0it} \leq T, \\
\sum_{i=1}^{T} w_{lt} = 1, \quad l \in \{ L_k \}, \\
w_{lt} \leq w_{0it}, \quad \forall t, \quad l \in \{ L_k \}, \\
w_{lt} = 0 \text{ or } 1, \quad l \in 0_k \cup \{ L_k \}, \forall t,
\]

(11)

(12)

Constraints (11) and (12) incorporate the fleet size and mix problem into the generalized assignment approach. (11) states that the number of vehicles cannot exceed some upper limit. (12) forces the variable \( w_{0it} = 1 \) if any other variable \( w_{lt} = 1 \) for the same \( i \) (i.e. if a truck services a customer, then the fixed cost must be incurred). Also if all \( w_{lt} = 0 \) for some \( i \), then no fixed cost should be incurred. Our formulation still can be separated into a non-linear assignment problem and a travelling salesman problem, as in Fisher and Jaikumar (1981):

Step 2a. Solve the assignment problem. For each open warehouse \( k \) i.e. \( z_k = 1 \): Define the set \( L_k = \{ l : \gamma_{kl} = 1 \} \) and \( n = \sum_{l \in L_k} \gamma_{kl} \).

\[
\min \sum_{i=1}^{T} f_i w_{0it} + \sum_{i=1}^{T} \sum_{m=1}^{n} d_{im} w_{imt}, \\
\text{s.t. } \sum_{i=1}^{n} d_i w_{it} \leq b_i, \forall t, \\
\sum_{t=1}^{T} w_{0it} \leq T, \\
\sum_{t=1}^{T} w_{lt} = 1, \quad l \in \{ L_k \}, \\
w_{lt} \leq w_{0it}, \quad \forall t, \quad l \in \{ L_k \}, \\
w_{lt} = 0 \text{ or } 1, \quad l \in 0_k \cup \{ L_k \}, \forall t.
\]

d_{im} is the cost of inserting customer \( i \) into vehicle route \( t \) in which vehicle \( t \) travels from the depot \( 0_k \) to \( l_t \), the seed customer of vehicle \( t \), and back to depot \( 0_k \). This cost is:

\[
d_{im} = \min \left( c_{0it} + c_{lt} + c_{0ilt} + c_{0itd} + c_{lt} + c_{0itd} \right)
\]

Since we are dealing now with the fleet size and mix problem and we wish to have the model indicate what kind of vehicle to choose, seed selection becomes more complicated. The number of seed customers which must be chosen for each vehicle type, is \( \left[ \sum_{i} d_i / b_i \right] \) for \( l \in \{ L_k \} \). The problem size becomes very great if the vehicle capacity is small and customer demand is large. Suppose total demand is 1000 and there are three truck types with capacities 100, 200 and 250. One must respectively choose 10, 5 and 4 seed customers, forcing the index \( t \) to denote 19 vehicle routes. However, different truck types may have the same
seed customer. These choices are made interactively by the model user, enabling selection of 'natural' seeds (e.g. customers whose demand is greater than half the capacity of a truck).

Let \( W_k = \{ w: w_{ik} = 1, \forall i, t \} \) be the optimal solution to the assignment problem. This solution determines the vehicle capacity and the assignment of customers to routes.

Step 2b. Solve the travelling salesman problem for each vehicle route \( t \) for which \( w_{0(t)} = 1 \). The solution to Step 2a gives the vehicle type used in servicing each customer.

Define \( n = \sum w_{it} \) and set \( \mu_{kt} = \{ i: w_{it} = 1 \} \).

\[
\begin{align*}
\min & \sum_{i=1}^{n} \sum_{m=1}^{T} \sum_{t=1}^{T} c_{lim} u_{lmt} \\
+ & \sum_{m=1}^{n} \sum_{l=1}^{T} f_{l} w_{0(mlt)},
\end{align*}
\]

s.t.
\[
\begin{align*}
\sum_{i=1}^{n} u_{lmt} &= w_{mt}, \quad m \in k \cup \{ L_{kt} \}, \\
\sum_{m=1}^{n} u_{lmt} &= w_{lt}, \quad l \in k \cup \{ L_{kt} \}, \\
\sum_{l \in S} \sum_{m \in S} u_{lmt} &\leq |S| - 1, \\
2 \leq |S| &\leq n - 1, \quad S \subseteq \{ L_{kt} \}, \\
v_{lmt} &\geq 0 \text{ or } 1, \quad l, m \in 0 \cup \{ L_{kt} \}.
\end{align*}
\]

Let \( V_{kt} = \{ v: v_{imt} = 1, \forall i, m, t \} \) be the optimal solution to the travelling salesman problem.

Step 2c. The routing solution to the fleet size and mix problem will indicate, for each vehicle route \( t \), the sequence in which that vehicle serves those customers. One thus knows the fixed cost \( f_{l} \) and the variable arc costs to traverse this route.

To compute the outbound transportation costs, \( c_{ikt} \), the following variables: \( f_{l} \), \( v_{lmt} \), and \( w_{0(mlt)} \) are used.

\( f_{l} \) is the distributed fixed cost of truck \( t \) per unit demand to customer \( l \) as a result of that truck serving that customer. Define \( t(l) \) as the \( t \)-index for which \( w_{lt} = 1 \); then,

\[
f_{l} = \frac{f_{l(i)}}{\sum_{m=1}^{T} w_{mt} D_{il}}.
\]

We wish to determine the cost of servicing each customer. That cost is approximately the average cost of the two arcs involving this customer, but because each vehicle route has one more arc than customers, one must also account for the arcs emanating from the warehouse. Those arc costs will be distributed to all customers on the route as \( f_{s} \), the 'fixed-route' cost per unit demand:

\[
f_{s} = \frac{\sum_{m=1}^{n} v_{m0(t)} c_{0ml(t)} + v_{0mt} c_{0mt(l)}}{2 \sum_{m=1}^{n} w_{mt} D_{il}}.
\]

The cost of servicing customer \( l \) per unit demand is thus

\[
c_{lt} = \frac{\sum_{m=1}^{n} v_{m0(t)} c_{0ml} + v_{0mt} c_{0mt(l)}}{2 \sum_{m=1}^{n} D_{il}}.
\]

The outbound costs \( c_{ikt} = f_{l} + f_{s} + c_{ikt, l} \) and the transportation costs are then \( c_{ikt} = c_{ikt} + c_{ikt, l} \).

Step 3a. Solve the transportation sub-problems \( \forall \) products \( i \):

\[
\begin{align*}
\min & \sum_{j} c_{jkt} x_{jkt}, \\
\text{s.t.} & \sum_{j} x_{jkt} \leq S_{j}, \quad \forall j, \\
& \sum_{j} x_{jkt} = D_{lt}, \quad \forall l, \\
x_{jkt} &\geq 0, \quad \forall j, l.
\end{align*}
\]

where \( k(l) \) is defined for each \( l \) as the \( k \)-index for which \( y_{kl} = 1 \) in the temporary fixed \( y \) array. Let the optimal solution be \( (x^{H+1}) \) and the optimal value be \( T(y^{H+1}) \).

Set

\[
UB^{H+1} = T(y^{H+1}) + \sum_{k} F_{k}^{H+1} y_{k}^{H+1} + \sum_{l} \sum_{k} V_{kl} D_{lt} y_{k}^{H+1}.
\]

The upper bound is the actual cost of the distribution system, the sum of the warehouse costs and transportation costs.

Step 3b. Determine the optimal dual solution to (13) with \( y = y^{H+1} \) and denote it by \( u^{H+1} \) and \( p^{H+1} \) corresponding to the supply and demand constraints (14) and (15) respectively.

Since the transportation problem breaks into \( i \) independent problems, the relationship between
the dual variables of the latter problems and those of the former problem must be known. That relationship between the variables is as follows (Geoffrion and Graves, 1974). If we denote the dual variables of the independent transportation problems as \( s_{ij} \) and \( d_{il} \) corresponding to the supply and demand constraints respectively, then:

\[
u_{ij} = s_{ij}, \quad \forall i, j,
\]

for \( k = k(l) \), \( P_{ikl} = d_{il}, \quad \forall i, l,
\]

and for \( k \neq k(l) \), \( P_{ikl} = \min(c_{ijkl} - u_{ij}) \)

for \( \forall i, l \).

The issue is now the estimation of \( c_{ijkl} \) for \( k \neq k(l) \). The first two terms of this cost (\( c_{ij} \) and \( c_{ijj} \)) are easily determined. However, calculating the costs of servicing each customer from a warehouse to which that customer is not presently assigned is difficult. Some method had to be derived to estimate those costs, with computational burden far less than required to work out routes for these alternative assignments.

On several test problems, formulae were attempted to estimate costs \( c_{ij} \) for which \( k \neq k(l) \). Of the 10 formulae tried (Reece, 1985), Formulae 1 and 2 (see Table 1) were best, solving the test problems to within 2% of optimality. Formula 2 was finally selected. We felt it would perform better on a wider variety of problems, and it also worked better when employing Technique II (see Section 5.2).

Therefore, for \( k \neq k(l) \), the outbound cost is

\[
c_{ikl} = a_{ikl} \star \sum B_l D_H
\]

\[
+ \frac{c_{0l} + c_{q} - cu_{t(l)} \star \sum B_l D_H}{2 \sum D_H}
\]

and the dual variable for the demand constraint is

\[
P_{ikl} = \min(c_{ij} + c_{ijk} + c_{ikl} - u_{ij}), \quad \forall i, l,
\]

where \( q \) = farthest customer from customer \( l \), \( cu_{t(l)} \) = cost/unit distance of vehicle \( t(l) \) and \( a_{ikl} \) = cost/unit volume shipped on route \( t(l) \). (It is assumed that the vehicle type is the same as that currently servicing customer \( l \)).

Step 3c. Since the dual variables are only cost estimates, it was found that the lower bound did not improve at every iteration. If the costs \( c_{ikl} \) for \( k \neq k(l) \) were overestimated, the lower bound would improve but if these costs were underestimated, that bound would decrease. In the typical Benders decomposition, the procedure is stopped when the lower bound and upper bound are within a few percent of each other. Our lower bound is merely an estimate that helps find the least cost solution; it is not a good indication of solution convergence.

Therefore, our stopping rule was taken as \( UB^{H+1} = UB^H \), i.e. when there is no further improvement in the upper bound. Otherwise go back to Step 1. The minimum upper bound is denoted \( UB^* \). If \( UB^{H+1} < UB^* \), then \( UB^* = UB^{H+1} \). Any solutions for which \( UB > UB^* \) are eliminated by adding cutting plane constraints that do not allow that solution to appear again.

Each time the transportation problems are solved, the dual variables provide an additional constraint for the warehouse problem. This constraint for \( h = 1, \ldots, H \) is given by:

\[
- Y_0 + \sum \sum \sum \sum \sum \sum P_{ikl} D_H Y_{ikl} + \sum \sum \sum u_{ij} S_{ij} \leq 0.
\]

The last step is to increment \( H \) by 1.

5. Computational experience

Steps 0 through 3c describe, for our model formulation, the solution algorithm of Figure 2 (referred to as "Technique I"). Based upon computational experience which we shall now describe, a more involved algorithm (Technique II) is presented in Section 5.2. Technique II would be employed at the user's discretion, depending upon results of some iterations using Technique I.

### Table 1

<table>
<thead>
<tr>
<th>No.</th>
<th>Formula</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{2c_{0l}}{\sum D_H} )</td>
<td>Twice the cost of the arc from warehouse to customer ( l ). Cost of a vehicle travelling from warehouse to customer and back</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{c_{0l} + c_{q}}{2 \sum D_H} )</td>
<td>Average cost of arcs from warehouse to customer ( l ) and from customer ( l ) to customer ( q ) where ( q ) is the farthest customer from customer ( l )</td>
</tr>
</tbody>
</table>
5.1. Discussion of test problems

To initially test the solution procedure, data for eleven problems was randomly generated. All eleven test problems had 2 products (of differing unit volumes) and 2 plants. Each problem had either 5 or 6 customers, at random locations. There were always 4 candidate warehouses (distribution centres), from which up to 2 were chosen. Warehouses generally differed in their capacities and fixed costs, and in the variable cost for each product. There were always 2 truck types with distinct capacities, fixed and variable costs. Inbound transportation costs at each warehouse were naturally dependent on the location of that facility and on the particular factory involved.

Feasible solutions for the small test problems were easily determined and the solution procedure began at Step 2. Because the optimal solution could be found by complete enumeration, it was possible to study the number of iterations required to attain optimality. Generally this was < 3 iterations. However for problems 5, 7, and 8, the final solution obtained on the first attempt (with Technique I) was not always optimal. Therefore several different starting feasible solutions were tried. For problems 5 and 7, the optimal solution was found for one particular starting solution, while the optimal solution was never found for problem 8. However, in all three problems, the worst final solution from Technique I differed by less than 2% from optimality, an acceptable difference for many applications.

The particular starting solution was thus an important factor for each problem. Varying the starting solution often led to a different final solution and required a different number of iterations to get there. A summary of results with Technique I for problems 5, 7 and 8 is given in Table 2.

### Table 2

| Problem no. | Number of starting sol'n attempted | Frequency of no. iterations to final solution (%) | Number of optimal solutions found | Worst sol'n
<table>
<thead>
<tr>
<th></th>
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<td>7</td>
<td>14 29 43 14 0</td>
<td>0.28</td>
<td></td>
</tr>
</tbody>
</table>

* A full description of the problems is given in Reece (1985).  
\( b \) Percentage away from optimality for the worst case.

**Step 3c'.** The upper bound is the cost of the current solution: the warehouse costs plus actual transportation costs. Going from one iteration to the next sometimes caused that feasible solution to become worse. That is, the upper bound may increase. The costs of assigning customers to warehouses (used in the warehouse problem) are only estimates, so assigning customers to other warehouses may increase transportation costs.

After solving the transportation problems, the dual variables are obtained and estimates \( y_0 \) of the transportation costs are used to solve the warehouse problem. Following the solution to that problem and then the fleet size and mix problem, the actual transportation costs are obtained. Comparing these actual costs to their estimate \( y_0 \) used in finding the solution gives the inaccuracy of estimating \( c_{ijkl} \) for \( k \neq k(l) \).

The difference between actual and estimated costs is calculated, and used to adjust the estimate whenever the upper bound is larger than the minimum upper bound. Each time a new minimum upper bound is found, a new dual constraint is generated. Denote the iteration number by \( H^* \) for which the minimum upper bound was found, and that bound by \( UB^* \). We introduce new parameters \( A_{H^*} \) as adjustment factors for the coefficients \( c_{ijkl}^* \) in the dual constraint.

At the current iteration \( H + 1 \), the solution to the warehouse problem is given by the vectors \( y_{H+1}^* \) and \( z_{H+1}^* \). The routing problems and transportation problems are then solved. The latter generate the dual variables \( P_{sk} \) for \( k = k(l) \), the actual transportation costs. At this stage the upper bound is obtained. The coefficients in the dual constraint are adjusted if \( UB^{H+1} > UB^* \) accord-
ing to the following recursive formulae for \( k = k(l) \).

\[
A_{kl}^{H+1} = A_{kl}^H + C_{kl}^{H+1} - C_{kl}^H,
\]
\[
C_{kl}^{H+1} = C_{kl}^H + A_{kl}^{H+1}.
\]

The dual variables for \( k = k(l) \) and the coefficients for the dual constraint are determined as in Technique I. The formula for the coefficients is

\[
C_{kl} = \sum_q D_{kl} Q_{kl}, \forall k, l.
\]

If UB\(^{H+1} \) < UB\(^* \), replace H\(^* \) with the current iteration number and UB\(^* \) with UB\(^{H+1} \). Increase H by 1. Initialize A\(_{kl} \) to zero.

**Step 3d':** Again all solutions for which UB > UB\(^* \) are eliminated by adding a cutting plane constraint. When UB\(^{H+1} = UB^H \), stop. Otherwise go to Step 1.

Incorporating 3c' and 3d' in Solution Technique II produced optimal solutions for all eleven test problems. To estimate the cost of servicing a customer from all warehouses, we used Formula 2 (Table 1). In most problems, the number of iterations increased by only one or two over Technique I (see Table 3).

Test problem 8 was more difficult. Technique I produced no optimal solution, even after trying seven different starting solutions. The optimal solution was obtained using Technique II. Eleven iterations were necessary, however this number is small compared to the 110 feasible solutions with only two distribution centres open. The model user must decide whether attaining optimality warrants the increased computational time of Technique II. The decision to use that technique might be made interactively, depending upon the first few iterations of Technique I and the sequence of lower bounds.

Technique II required more iterations when there were many feasible solutions near the optimal solution, compared to when there were few. For a particular problem using Technique II, four out of ten feasible solutions found were less than 1% away from optimality. A local minimum may have prevented the global optimum from being obtained using Technique I and made the problem harder to solve with Technique II. Also, when a customer location moved further away from the main cluster of customers and warehouses, an increased number of iterations was required.

A greater number of decision variables of course increases the number of columns in the problem, but it also indirectly increases the number of rows. It is the extra constraints, especially those concerning additional customers, which make the problem harder to solve. Table 4 indicates the number of variables and constraints in the linear programming sub-problems for a particular iteration. The example has 12 customers, 4 depots, 3 products and 3 plants. If the number of customers increases to 16, the number of variables and constraints increase (figures in brackets). Further testing is needed to determine what other features make a given problem easy or difficult.

<table>
<thead>
<tr>
<th>Problem no.</th>
<th>Solution Technique I</th>
<th>Solution Technique II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iterations</td>
<td>Optimal solution?</td>
<td>Warehouse</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>yes</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>yes</td>
</tr>
<tr>
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<td>4</td>
<td>yes</td>
</tr>
<tr>
<td>5</td>
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<td>6</td>
<td>4</td>
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</tr>
<tr>
<td>7</td>
<td>3</td>
<td>yes</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
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</tr>
<tr>
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</tr>
<tr>
<td>10</td>
<td>2</td>
<td>yes</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>yes</td>
</tr>
</tbody>
</table>

\(^{a}\) Optimal solution obtained for all problems.

\(^{b}\) Each iteration involves solving a warehouse problem, a routing problem and a transportation problem.

\(^{c}\) Number of iterations (average over 8 different starting solutions)

\(^{d}\) Number of iterations (average over 5 different starting solutions)

\(^{e}\) Number of iterations (average over 7 different starting solutions)

\(^{f}\) These problems had only 3 possible warehouse locations, rather than 4.

<table>
<thead>
<tr>
<th>Program module</th>
<th>Number of inequality constraints</th>
<th>Number of equality constraints</th>
<th>Number of variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Warehouse</td>
<td>48 (64)</td>
<td>12 (16)</td>
<td>53 (69)</td>
</tr>
<tr>
<td>Assign</td>
<td>22 (28)</td>
<td>7 (9)</td>
<td>21 (27)</td>
</tr>
<tr>
<td>Sails</td>
<td>0 (0)</td>
<td>8 (10)</td>
<td>8 (10)</td>
</tr>
<tr>
<td>Transp</td>
<td>3 (3)</td>
<td>12 (16)</td>
<td>36 (48)</td>
</tr>
</tbody>
</table>

\(^{a}\) Not including sub-tour breaking, cutting plane, dual or additional configuration constraints.
6. Suggestions for further research

For our sample test problems, the final solutions using Technique I were within 2% of optimality and final solutions using Technique II were optimal. However, for larger problems where the optimal solution is not known, one needs to determine how close a final solution is to optimality. An analytical worst-case study might be attempted. Probabilistic analysis could establish some statistical properties of the algorithms.

Greater payoff may be available in determining a sharper lower bound. Also, to determine a lower bound which will improve at each iteration, we require a method better than our Formula 2 to estimate the outbound costs for alternative customer—warehouse pairs.

Other areas for additional research pertain to algorithms for the warehouse or the vehicle routing problems. For the warehouse problem, one could investigate improved cutting plane algorithms or the use of Lagrangian relaxation (Christofides and Beasley, 1983). The generalised assignment approach to the fleet size and mix vehicle routing problem should be tested on larger problems, for which the number of variables in the travelling salesman sub-problem and the assignment sub-problem increase dramatically.

References


